

EQUIVARIANT COHOMOLOGY OF MODULI SPACES OF GENUS THREE CURVES WITH LEVEL TWO STRUCTURE

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ABSTRACT. We compute cohomology of the moduli space of genus three curves with level two structure and some related spaces. In particular, we determine the cohomology groups of the moduli space of plane quartics with level two structure as representations of the symplectic group on a six dimensional vector space over the field of two elements. We also make the analogous computations for some related spaces such as moduli spaces of genus three curves with a marked points and strata of the moduli space of Abelian differentials of genus three.

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1. INTRODUCTION

The purpose of this note is to compute the de Rham cohomology (with rational coefficients) of various moduli spaces of curves of genus 3 with level 2 structure. The group $\mathrm{Sp}(6, \mathbb{F}_2)$ acts on the set of level 2 structures of a curve. This action induces actions on the various moduli spaces which in turn yields actions on

the cohomology groups which thus become $\mathrm{Sp}(6, \mathbb{F}_2)$ -representations and our goal is to describe these representations.

The moduli spaces under consideration are essentially of three different types. First of all, we have the moduli space $\mathcal{M}_3[2]$ of genus 3 curves with level 2 structure and some natural loci therein. This will be our main object of study. Secondly we will consider the moduli space $\mathcal{M}_{3,1}[2]$ of genus 3 curves with level 2 structure and one marked point and some of its subspaces. Thirdly, we have the moduli space $\mathcal{H}\mathcal{O}\mathcal{L}_3[2]$ of genus 3 curves with level 2 structure marked with a holomorphic (i.e. Abelian) differential and some related spaces, e.g. the moduli space of genus 3 curves marked with a canonical divisor. There are many constructions, some classical and some new, relating the various spaces and which will provide essential information for our cohomological computations.

The plan of the paper is as follows. In Section 2 we give the basic definitions and sketch some of the classical theory around genus 3 curves and their level 2 structures. In Section 3 we sketch a construction, due to Looijenga [Loo93], which expresses some natural loci in $\mathcal{M}_{3,1}$ in terms of arrangements of tori and hyperplanes and we use this description to compute the cohomology of these loci. Hyperelliptic curves will be somewhat peripheral in this note but we give a discussion in Section 4. In Section 5 we recall some constructions and results regarding strata of moduli spaces of Abelian differentials, essentially due to Kontsevich and Zorich [KZ03], and we make cohomological computations of these strata in genus 3. The core of the paper is Section 6 where we compute the cohomology of the moduli space $\mathcal{Q}[2]$ of plane quartics with level 2 structure as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$. Finally, in Section 7 we make some comments around the cohomology of $\mathcal{M}_3[2]$.

The main results of this note are presented in Section 8 in the form of tables. However, for convenience (and for readers not interested in the representation structure of the cohomology) we also give some results in the form of Poincaré polynomials, for instance in Theorems 3.4, 3.5 and 4.2.

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2. BACKGROUND

We work over the field of complex numbers.

2.1. Level structures. Let C be a smooth and irreducible curve of genus g and let $\mathrm{Jac}(C)$ denote its Jacobian. For any positive integer n there is an isomorphism

$$\mathrm{Jac}(C)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g},$$

where $\mathrm{Jac}(C)[n]$ denotes the subgroup of n -torsion elements in $\mathrm{Jac}(C)$. A *symplectic level n structure* on C is an ordered basis (D_1, \dots, D_{2g}) of $\mathrm{Jac}(C)[n]$ such that the Weil pairing has matrix of the form

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

with respect to this basis, where I_g denotes the identity matrix of size $g \times g$. We will often drop the adjective “symplectic” and simply say “level n structure”. There is a moduli space of curves of genus g with level n structure which we denote by $\mathcal{M}_g[n]$. For $n \geq 3$ it is fine but not for $n = 2$ since a level 2 structure on a hyperelliptic curve is preserved by the hyperelliptic involution. The symplectic group $\mathrm{Sp}(2g, \mathbb{Z}/n\mathbb{Z})$ acts on $\mathcal{M}_g[n]$ by changing the level n structure.

2.2. Curves of genus three. Suppose that C is of genus 3. If C is not hyperelliptic, then a choice of basis of its space of global sections gives an embedding of C into the projective plane as a curve of degree 4. As is easily seen via the genus-degree formula, every smooth plane quartic curve is of genus 3 and we thus have a decomposition

$$\mathcal{M}_3[n] = \mathcal{Q}[n] \sqcup \mathcal{Hyp}_3[n],$$

where $\mathcal{Q}[n]$ denotes the *quartic locus* and $\mathcal{Hyp}_3[n]$ denotes the *hyperelliptic locus*.

From now on we shall specialize to the case $n = 2$. The locus $\mathcal{Q}[2]$ is by far the more complicated of the two loci and its investigation will therefore take up most of this note, but we will also consider hyperelliptic curves in Section 4.

There is a close relationship between level 2 structures on a plane quartic and its bitangents. More precisely, if $C \in \mathbb{P}^2$ is a plane quartic curve and $B \in \mathbb{P}^2$ is a bitangent line of C , then $C \cdot B = 2P + 2Q$ for some points P and Q on C . Thus, if we set $D = P + Q$ then $2D = K_C$. Divisors D with the property that $2D = K_C$ are called *theta characteristics*. A theta characteristic D is called *even* or *odd* depending on whether $h^0(D)$ is even or odd and it can be shown that there is a bijective correspondence between the set of odd theta characteristics of C and the set of bitangents of C given by the construction above.

Given two theta characteristics D and D' we obtain a 2-torsion element by taking the difference $D - D'$. This gives the set Θ of theta characteristics on C the structure of a $\mathrm{Jac}(C)[2]$ -torsor. The union $V = \mathrm{Jac}(C)[2] \sqcup \Theta$ is thus a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ of dimension 7. Alternatively, we can describe V as the 2-torsion subgroup of $\mathrm{Pic}(C)/\mathbb{Z}K_C$.

An ordered basis θ of V consisting of odd theta characteristics is called an *ordered Aronhold basis* if it has the property that the expression

$$h(D) \pmod{2}$$

only depends on the number of elements in θ required to express D for any theta characteristic D .

Proposition 2.1. *There is a bijection between the set of ordered Aronhold bases on C and the set of level 2 structures on C .*

For a proof, see [DO88] or [GH04].

Thus, we may think of a level 2 structure on C as an ordered Aronhold basis of odd theta characteristics on C . Since odd theta characteristics are cut out by bitangents we can also think about level 2 structures in terms of ordered sets of seven bitangents (but we must then bear in mind that not every ordered set of seven bitangents corresponds to a level 2 structure).

2.3. Point configurations in the projective plane. Let P_1, \dots, P_7 be seven points in \mathbb{P}^2 . We say that the points are in *general position* if there is no “unexpected” curve passing through any subset of them, i.e. if

- no three of the points lie on a line and
- no six of the points lie on a conic.

We denote the moduli space of ordered septuples of points in general position in \mathbb{P}^2 up to projective equivalence by \mathcal{P}_7^2 .

Given seven points in general position in \mathbb{P}^2 there is a net \mathcal{N} of cubics passing through the points. The set of singular members of \mathcal{N} is a plane curve T of degree 12 with 24 cusps and 28 nodes. The dual $T^\vee \subset \mathcal{N}^\vee \cong \mathbb{P}^2$ is a smooth plane quartic curve. Another way to obtain a genus 3 is by taking the set S of singular points of members of \mathcal{N} . The set S is a sextic curve with ordinary double points precisely at P_1, \dots, P_7 . From this information it is easy to see, via the genus-degree formula, that S has geometric genus 3. One can also show that the map σ sending a point P in S to the unique member of \mathcal{N} with a singularity at P is a birational isomorphism from S to T .

2.4. Del Pezzo surfaces of degree two. Recall that a *Del Pezzo surface* is a smooth and projective algebraic variety of dimension two such that its anticanonical class is ample. The *degree* of a Del Pezzo surface S is the self intersection number of its canonical class, K_S^2 .

Given seven points $P = (P_1, \dots, P_7)$ in general position in \mathbb{P}^2 , the blow-up $X = \text{Bl}_P \mathbb{P}^2$ is a Del Pezzo surface of degree 2. Moreover, every Del Pezzo surface of degree 2 can be realized as such a blow-up, see [Man74]. We denote the blow-up map by $\pi : X \rightarrow \mathbb{P}^2$. However, the points P_1, \dots, P_7 do only give us the Del Pezzo surface X - we also get the seven exceptional curves E_1, \dots, E_7 . Together with the strict transform L of a line in \mathbb{P}^2 they determine a basis for the Picard group of X

$$\text{Pic}(X) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \dots \oplus \mathbb{Z}E_7.$$

The intersection theory is given by

$$L^2 = 1, \quad E_i^2 = -1, \quad L \cdot E_i = E_i \cdot E_j = 0, \quad i \neq j.$$

Not every ordered basis of $\text{Pic}(X)$ comes from a blow-up as above. Bases which do arise in this way are called *geometric markings*. Two geometrically marked Del Pezzo surfaces (X, E_1, \dots, E_7) and (X', E'_1, \dots, E'_7) are isomorphic if there is an isomorphism of surfaces $\phi : X \rightarrow X'$ such that $\phi^*(E'_i) = E_i$ for all i . We denote the moduli space of geometrically marked Del Pezzo surfaces of degree 2 by $\mathcal{DP}_2^{\text{gm}}$.

Given a quartic $C \subset \mathbb{P}^2$ we can obtain a Del Pezzo surface X of degree 2 as the double cover of \mathbb{P}^2 ramified along C . Moreover, every Del Pezzo surface of degree 2 can be realized as such a double cover, see [Kol96]. We let $p : X \rightarrow \mathbb{P}^2$ denote the covering map and let ι denote the involution exchanging the two sheets. If E_1, \dots, E_7 is a geometric marking of X then $p(E_1), \dots, p(E_7)$ is an ordered Aronhold set of bitangents of C .

We have thus seen how to obtain a geometrically marked Del Pezzo surface of degree 2 both from seven ordered points in general position and from a plane quartic curve with level 2 structure and we have also seen how to obtain the quartic curve directly from the seven points. We summarize the situation in the diagram below.

$$\begin{array}{ccc} & X & \\ \pi \swarrow & & \searrow p \\ \mathbb{P}^2_{\text{pts}} & \overset{\sigma^\vee}{\dashrightarrow} & \mathbb{P}^2_{\text{curve}} \end{array}$$

Here σ^\vee denotes the composition of σ and the dualization map. In each of the spaces we have a copy of the curve C : in $\mathbb{P}_{\text{curve}}^2$ we have the actual curve C , in X we obtain an isomorphic copy of C by taking the fixed locus of the involution ι and in $\mathbb{P}_{\text{pts}}^2$ we have a sextic model S of C with seven double points.

Theorem 2.2 (van Geemen, [DO88]). *The above construction yields $\text{Sp}(6, \mathbb{F}_2)$ -equivariant isomorphisms*

$$\begin{array}{ccc} & \mathcal{DP}_2^{\text{gm}} & \\ \swarrow & & \searrow \\ \mathcal{P}_7^2 & \longleftrightarrow & \mathcal{Q}[2] \end{array}$$

3. CURVES AND SURFACES WITH MARKED POINTS

3.1. Genus three curves with marked points. We now turn our attention to the moduli space $\mathcal{M}_{3,1}[2]$ of genus 3 curves with level 2 structure and one marked point. Also in this case we have a decomposition

$$\mathcal{M}_{3,1}[2] = \mathcal{Q}_1[2] \sqcup \mathcal{Hyp}_{3,1}[2]$$

into a quartic locus and a hyperelliptic locus. However, in this case there is also a natural decomposition of the quartic locus in terms of the behaviour of the tangent line at the marked point.

Let C be a plane quartic curve, let P be a point on C and let $T_P \subset \mathbb{P}^2$ denote the tangent line of C at P . Since C is of degree 4, Bézout's theorem tells us that the intersection product $C \cdot T_P$ will consist of 4 points. There are four possibilities:

- (i) $T_P \cdot C = 2P + Q + R$ where Q and R are two distinct points on C , both different from P . In this case, T_P is called an *ordinary tangent line* of C and P is called an *ordinary point* of C .
- (ii) $T_P \cdot C = 2P + 2Q$ where $Q \neq P$ is a point on C . In this case, T_P is called a *bitangent* of C and P is called a *bitangent point* of C .
- (iii) $T_P \cdot C = 3P + Q$ where $Q \neq P$ is a point on C . In this case, T_P is called a *flex line* of C and P is called a *flex point* of C .
- (iv) $T_P \cdot C = 4P$. In this case, T_P is called a *hyperflex line* of C and P is called a *hyperflex point* of C .

This yields a decomposition of $\mathcal{Q}_1[2]$

$$\mathcal{Q}_1[2] = \mathcal{Q}_{\text{ord}}[2] \sqcup \mathcal{Q}_{\text{flx}}[2] \sqcup \mathcal{Q}_{\text{btg}}[2] \sqcup \mathcal{Q}_{\text{hfl}}[2]$$

into a locus of curves marked with an ordinary, flex, bitangent and hyperflex point, respectively.

3.2. Del Pezzo surfaces of degree two with marked points. Let X be a Del Pezzo surface of degree 2. Recall that we can realize S both as a double cover $p : S \rightarrow \mathbb{P}^2$ ramified over a plane quartic C and as the blowup $\pi : X \rightarrow \mathbb{P}^2$ in seven points P_1, \dots, P_7 in general position. Also recall that there is an involution ι of X and that we can identify the fixed points of ι in X with $p^{-1}(C)$. We shall now give another characterization of the fixed points of ι .

A curve $A \subset X$ in the anticanonical linear system $| -K_X |$ is called an *anti-canonical curve*. The anticanonical class $-K_X = 3L - E_1 - \dots - E_7$ corresponds to the linear system \mathcal{C} of cubics in \mathbb{P}^2 passing through P_1, \dots, P_7 . The curve $B = \pi(p^{-1}(C))$ consists of all the singular points of members of \mathcal{C} . We thus see

that a point $Q \in X$ is a point of $p^{-1}(C)$ if and only if there is a unique anticanonical curve A with a singularity at Q . Note that since A is isomorphic to a singular plane cubic, its irreducible components will be rational.

By the above construction we have that if (C, P) is a plane quartic with a marked point, the double cover $p : X \rightarrow \mathbb{P}^2$ ramified along C naturally becomes equipped with an anticanonical curve A with a singularity at the inverse image of P . Thus, if we introduce the moduli space $\mathcal{DP}_{2,a}^{\text{gm}}$ of geometrically marked Del Pezzo surfaces of degree 2 marked with a singular point of an anticanonical curve we have an isomorphism $\mathcal{Q}_1[2] \cong \mathcal{DP}_{2,a}^{\text{gm}}$.

Since $p^{-1}(C) \sim -2K_X$ it follows that

$$A.p^{-1}(C) = (-K_X).(-2K_X) = 4.$$

We have that A intersects $p^{-1}(C)$ with multiplicity at least 2 so $p(A)$ is a tangent to C . The anticanonical curve A can be of the following types.

- (i) The anticanonical curve A can be an irreducible curve with a node. Then $p(A)$ intersects C with multiplicity 2 at P so $p(A)$ is either an ordinary tangent line or a bitangent. But we have shown that the inverse image of a bitangent under p consists of two exceptional curves which are conjugate under ι and we conclude that $p(A)$ is an ordinary tangent line. We may thus identify the locus $\mathcal{DP}_{2,n}^{\text{gm}} \subset \mathcal{DP}_{2,a}^{\text{gm}}$ consisting of surfaces such that the anticanonical curve through the marked point is irreducible and nodal with the locus $\mathcal{Q}_{\text{ord}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is ordinary.
- (ii) The anticanonical curve A can be an irreducible curve with a cusp. Then $p(A)$ intersects C with multiplicity 3 at P so $p(A)$ must be a flex line. We may thus identify the locus $\mathcal{DP}_{2,c}^{\text{gm}} \subset \mathcal{DP}_{2,a}^{\text{gm}}$ consisting of surfaces such that the anticanonical curve through the marked point is irreducible and cuspidal with the locus $\mathcal{Q}_{\text{flx}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is a genuine flex point.
- (iii) The anticanonical curve A can consist of two rational curves intersecting with multiplicity one at P . Thus, the cubic $\pi(A)$ must be the product of a conic through five of the points P_1, \dots, P_7 with a line through the remaining two. Hence, A consists of two conjugate exceptional curves and $p(A)$ is a genuine bitangent. We may thus identify the locus $\mathcal{DP}_{2,t}^{\text{gm}} \subset \mathcal{DP}_{2,a}^{\text{gm}}$ consisting of surfaces such that the anticanonical curve through the marked point consists of two rational curves intersecting transversally in two distinct points with the locus $\mathcal{Q}_{\text{btg}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is a genuine bitangent point.
- (iv) The anticanonical curve A can consist of two rational curves intersecting with multiplicity two at P . An analysis similar to the one above shows that $p(A)$ is then a hyperflex line. We may thus identify the locus $\mathcal{DP}_{2,d}^{\text{gm}} \subset \mathcal{DP}_{2,a}^{\text{gm}}$ consisting of surfaces such that the anticanonical curve through the marked point consists of two rational curves with a double intersection with the locus $\mathcal{Q}_{\text{hfl}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is a hyperflex point.

In [Loo93], Looijenga gave descriptions of each of these loci in terms of arrangements. In order to give his results, we need to investigate the Picard group of X in a bit more detail.

The Del Pezzo surface X has exactly 56 exceptional curves which can be described as follows.

- (i) The 7 exceptional curves E_i .
- (ii) The 21 strict transforms of lines between two points P_i and P_j . The classes of these curves are given by $L - E_i - E_j$.
- (iii) The 21 strict transforms of conics through all but two points P_i and P_j . The classes of these curves are given by $2L - E_1 - \dots - E_7 + E_i + E_j$.
- (iv) The 7 cubics through P_1, \dots, P_7 with a singularity in one of the points P_i . The classes of these curves are given by $3L - E_1 - \dots - E_7 - E_i$.

We denote the set of these classes by \mathcal{E} .

The involution ι fixes the anticanonical class K_X . We denote the orthogonal complement of K_X in $\text{Pic}(X)$ by K_X^\perp . The involution ι acts as -1 on K_X^\perp . The elements α in K_X^\perp such that $\alpha^2 = -2$ form a root system Φ of type E_7 . We denote the Weyl group of E_7 by $W(E_7)$. The group $W(E_7)$ is isomorphic to $\text{Sp}(6, \mathbb{F}_2) \times \mathbb{Z}_2$ where \mathbb{Z}_2 is the group of two elements generated by ι . We denote the quotient $W(E_7)/\langle \iota \rangle$ by $W(E_7)^+$.

3.2.1. The irreducible nodal case. Let X be a geometrically marked Del Pezzo surface of degree 2 and let P be a point of X such that there is a unique rational anticanonical curve A on X which is nodal at P . The Jacobian $\text{Jac}(A)$ is isomorphic to k^* as a group, see Chapter II.6 of [Har77], and the restriction homomorphism

$$\text{Pic}(X) \rightarrow \text{Pic}(A),$$

induces a homomorphism

$$r : K_X^\perp \rightarrow \text{Jac}(A).$$

Recall that K_S^\perp is a lattice isometric to the E_7 -lattice L_{E_7} . We thus see that r is an element of the 7-dimensional algebraic torus $T = \text{Hom}(K_S^\perp, \text{Jac}(A)) \cong (k^*)^7$ and we have a natural action of $W(E_7)$ on T via its action on K_S^\perp .

Every root α in Φ determines a multiplicative character on T by evaluation, i.e. by sending an element $\chi \in T$ to $\chi(\alpha) \in k^*$. Let

$$T_\alpha = \{\chi \in T \mid \chi(\alpha) = 1\}$$

and define

$$D_{E_7} = \bigcup_{\alpha \in \Phi(E_7)} T_\alpha,$$

and let T_{E_7} be the complement $T \setminus D_{E_7}$. We remark that D_{E_7} is the toric arrangement associated to the root system E_7 .

Theorem 3.1 (Looijenga [Loo93]). *There is a $W(E_7)^+$ -equivariant isomorphism*

$$\mathcal{DP}_{2,n}^{\text{gm}} \rightarrow \{\pm 1\} \setminus T_{E_7}.$$

Since $\mathcal{Q}_{\text{ord}}[2]$ is isomorphic to $\mathcal{DP}_{2,n}^{\text{gm}}$ and $W(E_7)^+$ is isomorphic to $\text{Sp}(6, \mathbb{F}_2)$, it follows that there is a $\text{Sp}(6, \mathbb{F}_2)$ -equivariant isomorphism $\mathcal{Q}_{\text{ord}}[2] \cong \{\pm 1\} \setminus T_{E_7}$.

3.2.2. The other cases. The three other cases have similar descriptions. For instance, if we let V_{E_7} denote the complement of the hyperplane arrangement associated to E_7 we have the following.

Theorem 3.2 (Looijenga [Loo93]). *There is a $W(E_7)^+$ -equivariant isomorphism*

$$\mathcal{DP}_{2,c}^{\text{gm}} \rightarrow \mathbb{P}(V_{E_7}).$$

It follows that there is a $\mathrm{Sp}(6, \mathbb{F}_2)$ -equivariant isomorphism $\mathcal{Q}_{\mathrm{fix}}[2] \cong \mathbb{P}(V_{E_7})$.

In order to state the results for the remaining two cases we need to introduce a little bit of notation. Let E be an exceptional curve. Then $E + \iota(E) = K_X$. We denote the orthogonal complement of $\langle E, \iota(E) \rangle$ in $\mathrm{Pic}(X)$ by $\langle E, \iota(E) \rangle^\perp$. Since $K_X \in \langle E, \iota(E) \rangle$ we have $\langle E, \iota(E) \rangle^\perp \subset K_X^\perp$ and $\Phi_E = \Phi \cap \mathrm{Pic}_{\langle E, \iota(E) \rangle^\perp}^\perp(S)$ is a subroot system of type E_6 . We denote the Weyl group of E_6 by $W(E_6)$. We denote the complement of the toric arrangement associated to E_6 by T_{E_6} and we denote the complement of the hyperplane arrangement associated to E_6 by V_{E_6} . The elements of \mathcal{E} are in bijective correspondence with the cosets in the quotient $W(E_7)/W(E_6)$ and for each $e \in \mathcal{E}$ we let $T_{E_6}(e)$ be an isomorphic copy of T_{E_6} . similarly, we let $\mathbb{P}(V_{E_6})(e)$ be an isomorphic copy of $\mathbb{P}(V_{E_6})$. We then have the following two results.

Theorem 3.3 (Looijenga [Loo93]). *There are $W(E_7)^+$ -equivariant isomorphisms*

$$\mathcal{DP}_{2,t}^{\mathrm{gm}} \rightarrow \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} T_{E_6}(e)$$

and

$$\mathcal{DP}_{2,d}^{\mathrm{gm}} \rightarrow \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} \mathbb{P}(V_{E_6})(e).$$

It follows that there are $\mathrm{Sp}(6, \mathbb{F}_2)$ -equivariant isomorphisms $\mathcal{Q}_{\mathrm{btg}}[2] \cong \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} T_{E_6}(e)$ and $\mathcal{Q}_{\mathrm{hfl}}[2] \cong \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} \mathbb{P}(V_{E_6})(e)$.

3.3. Cohomological computations. We have thus seen how each of the four strata of $\mathcal{Q}_1[2]$ either can be described in terms of complements of toric arrangements or in terms of complements hyperplane arrangements. In the affine hyperplane case, the necessary computations were carried out by Fleischmann and Janiszczak, see [FJ93]. They present their results in terms of equivariant Poincaré polynomials and one goes from the affine to the projective case by dividing their results by $1+t$. In the toric case, the necessary computations were carried out by the author in [Ber16b].

Since $W(E_7) = \mathrm{Sp}(6, \mathbb{F}_2) \times \{\pm 1\}$ we have that each representation of $W(E_7)$ either is a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$ times the trivial representation of $\{\pm 1\}$ or a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$ times the alternating representation of $\{\pm 1\}$. Thus, to go from the cohomology of the complement of an arrangement associated to E_7 one simply takes the $\{\pm 1\}$ -invariant part. This explains how we obtained the cohomology of $\mathcal{Q}_{\mathrm{ord}}[2]$ and $\mathcal{Q}_{\mathrm{fix}}[2]$ given in Table 1 and Table 2, respectively. If one is only interested in the dimensions of the various cohomology groups, these are more conveniently given as Poincaré polynomials.

Theorem 3.4. *The cohomology groups $H^i(\mathcal{Q}_{\mathrm{ord}}[2])$ and $H^i(\mathcal{Q}_{\mathrm{fix}}[2])$ are both pure of Tate type (i, i) . The Poincaré polynomial of $\mathcal{Q}_{\mathrm{ord}}[2]$ is*

$$\begin{aligned} P(\mathcal{Q}_{\mathrm{ord}}[2], t) = & 1 + 63t + 1638t^2 + 22680t^3 + 180089t^4 + \\ & + 820827t^5 + 2004512t^6 + 2064430t^7 \end{aligned}$$

and the Poincaré polynomial of $\mathcal{Q}_{\mathrm{fix}}[2]$ is

$$\begin{aligned} P(\mathcal{Q}_{\mathrm{fix}}[2], t) = & 1 + 62t + 1555t^2 + 20180t^3 + 142739t^4 + \\ & + 521198t^5 + 765765t^6. \end{aligned}$$

To obtain the cohomology of $\mathcal{Q}_{\text{btg}}[2]$ from the cohomology of T_{E_6} we first have to induce from $W(E_6)$ and then take $\{\pm 1\}$ -invariants. Thus

$$H^i(\mathcal{Q}_{\text{btg}}[2]) = \text{Ind}_{W(E_6)}^{W(E_7)} (H^i(T_{E_7}))^{\{\pm 1\}}.$$

The results are given in Table 3. In an entirely analogous way one obtains the cohomology of $\mathcal{Q}_{\text{hfl}}[2]$ from the cohomology of $\mathbb{P}(V_{E_7})$. The results are given in Table 4.

Theorem 3.5. *The cohomology groups $H^i(\mathcal{Q}_{\text{btg}}[2])$ and $H^i(\mathcal{Q}_{\text{hfl}}[2])$ are both pure of Tate type (i, i) . The Poincaré polynomial of $\mathcal{Q}_{\text{btg}}[2]$ is*

$$P(\mathcal{Q}_{\text{btg}}[2], t) = 28 + 1176t + 19740t^2 + 168560t^3 + 768852t^4 + 1774584t^5 + 1639540t^6$$

and the Poincaré polynomial of $\mathcal{Q}_{\text{hfl}}[2]$ is

$$P(\mathcal{Q}_{\text{hfl}}[2], t) = 28 + 980t + 13300t^2 + 87500t^3 + 278992t^4 + 344960t^5.$$

Let $\mathcal{Q}_{\text{ord}}[2]$ be the union of $\mathcal{Q}_{\text{ord}}[2]$ and $\mathcal{Q}_{\text{flx}}[2]$ inside $\mathcal{Q}_1[2]$. By Looijenga's results [Loo93] we have that there is a $\text{Sp}(6, \mathbb{F}_2)$ -equivariant short exact sequence of mixed Hodge structures

$$0 \rightarrow H^i(\mathcal{Q}_{\text{ord}}[2]) \rightarrow H^i(\mathcal{Q}_{\text{ord}}[2]) \rightarrow H^{i-1}(\mathcal{Q}_{\text{flx}}[2])(-1) \rightarrow 0.$$

Thus, $H^i(\mathcal{Q}_{\text{ord}}[2])$ is pure of Tate type (i, i) and we may easily deduce the structure as a $\text{Sp}(6, \mathbb{F}_2)$ -representation from Tables 1 and 2. The result is given in Table 5.

Theorem 3.6. *The cohomology group $H^i(\mathcal{Q}_{\text{ord}}[2])$ is pure of Tate type (i, i) and the Poincaré polynomial of $\mathcal{Q}_{\text{ord}}[2]$ is*

$$P(\mathcal{Q}_{\text{ord}}[2], t) = 1 + 62t + 1576t^2 + 21125t^3 + 159909t^4 + 678068t^5 + 1483314t^6 + 1302665t^7.$$

Similarly, let $\mathcal{Q}_{\text{btg}}[2]$ be the union of $\mathcal{Q}_{\text{btg}}[2]$ and $\mathcal{Q}_{\text{hfl}}[2]$ inside $\mathcal{Q}_1[2]$. Again, by results of Looijenga [Loo93] we have that there is a $\text{Sp}(6, \mathbb{F}_2)$ -equivariant short exact sequence of mixed Hodge structures

$$0 \rightarrow H^i(\mathcal{Q}_{\text{btg}}[2]) \rightarrow H^i(\mathcal{Q}_{\text{btg}}[2]) \rightarrow H^{i-1}(\mathcal{Q}_{\text{hfl}}[2])(-1) \rightarrow 0.$$

Thus, $H^i(\mathcal{Q}_{\text{btg}}[2])$ is pure of Tate type (i, i) and we may easily deduce the structure as a $\text{Sp}(6, \mathbb{F}_2)$ -representation from Tables 3 and 4. The result is given in Table 6.

Theorem 3.7. *The cohomology group $H^i(\mathcal{Q}_{\text{btg}}[2])$ is pure of Tate type (i, i) and the Poincaré polynomial of $\mathcal{Q}_{\text{btg}}[2]$ is*

$$P(\mathcal{Q}_{\text{btg}}[2], t) = 28 + 1148t + 18760t^2 + 155260t^3 + 681352t^4 + 1495592t^5 + 1294580t^6.$$

4. HYPERELLIPTIC CURVES

We shall now briefly turn our attention to the moduli spaces of hyperelliptic curves, $\mathcal{Hyp}_3[2]$ and $\mathcal{Hyp}_{3,1}[2]$.

Let C be a hyperelliptic curve of genus $g \geq 2$. Then C can be realized as a double cover of \mathbb{P}^1 ramified over $2g + 2$ points. Moreover, if we pick $2g + 2$ ordered points on \mathbb{P}^1 , the curve C obtained as the double cover ramified over precisely those points is a hyperelliptic curve and the points also determine a level 2-structure on

C . However, not all level 2-structures on C arise in this way. Nevertheless, there is an intimate relationship between the moduli space $\mathcal{Hyp}_g[2]$ and the moduli space $\mathcal{M}_{0,2g+2}$ of $2g+2$ ordered points on \mathbb{P}^1 .

Theorem 4.1 (Dolgachev and Ortland [DO88], Tsuyumine [Tsu91]). *Let \mathfrak{S} denote the set of left cosets $\mathrm{Sp}(2g, \mathbb{F}_2) / S_{2g+2}$, let $[\sigma] \in \mathfrak{S}$ and let $X_\sigma = \mathcal{M}_{0,2g+2}$. Then*

$$\mathcal{Hyp}_g[2] \cong \coprod_{[\sigma] \in \mathfrak{S}} X_{[\sigma]},$$

and $\mathrm{Sp}(2g, \mathbb{F}_2)$ acts transitively on the set of connected components of $\mathcal{Hyp}_g[2]$.

Thus, the cohomology of $\mathcal{Hyp}_g[2]$ can be obtained by computing the cohomology of $\mathcal{M}_{0,2g+2}$ as a S_{2g+2} -representation and then inducing up to $\mathrm{Sp}(2g, \mathbb{F}_2)$. More precisely, we have

$$H^i(\mathcal{Hyp}_g[2]) = \mathrm{Ind}_{S_{2g+2}}^{\mathrm{Sp}(2g, \mathbb{F}_2)} (H^i(\mathcal{M}_{0,2g+2}))$$

where we consider $H^i(\mathcal{M}_{0,2g+2})$ as a S_{2g+2} -representation and $H^i(\mathcal{Hyp}_g[2])$ as a $\mathrm{Sp}(2g, \mathbb{F}_2)$ -representation. One way to compute the cohomology of $\mathcal{M}_{0,2g+2}$ is to make S_{2g+2} -equivariant point counts of $\mathcal{M}_{0,2g+2}$. Since $\mathcal{M}_{0,2g+2}$ is isomorphic to a hyperplane arrangement, this will determine its cohomology, see [DL97]. For $\mathcal{Hyp}_3[2]$, these computations were carried out in [Ber16c] and the results are given, for convenience, in Table 10. We also mention that $H^k(\mathcal{Hyp}_3[2])$ is pure of Tate type (k, k) .

Theorem 4.2. *The Poincaré polynomial of $\mathcal{Hyp}_3[2]$ is*

$$P(\mathcal{Hyp}_3[2], t) = 36 + 720t + 5580t^2 + 20880t^3 + 37584t^4 + 25920t^5$$

The moduli space $\mathcal{Hyp}_{3,1}[2]$ (as a coarse moduli space) is a \mathbb{P}^1 -fibration over $\mathcal{Hyp}_3[2]$ via the forgetful morphism. The Leray-Serre spectral sequence of this fibration degenerates at the second page and allows us to compute the cohomology of $\mathcal{Hyp}_{3,1}[2]$, together with its mixed Hodge structure, as

$$H^k(\mathcal{Hyp}_{3,1}[2]) = H^{k-2}(\mathcal{Hyp}_3[2])(-1) \oplus H^k(\mathcal{Hyp}_3[2]).$$

Thus, the cohomology of $\mathcal{Hyp}_{3,1}[2]$ is easily obtained via Table 10.

5. MODULI OF ABELIAN DIFFERENTIALS

Let $\mathcal{H}\mathcal{O}\mathcal{L}_g$ denote the moduli space of pairs (C, ω) where C is a curve of genus g and ω is a nonzero holomorphic 1-form (i.e. an Abelian differential) and let $\mathcal{H}\mathcal{O}\mathcal{L}_g[2]$ denote the corresponding moduli space where the curves are also marked with a level 2 structure. Kontsevich and Zorich [KZ03] gave stratification of $\mathcal{H}\mathcal{O}\mathcal{L}_g[2]$ according to the multiplicities of the zeros of ω and we follow them in order to obtain a corresponding stratification of $\mathcal{H}\mathcal{O}\mathcal{L}_g[2]$. More precisely, let $\lambda = [\lambda_1, \dots, \lambda_n]$ be a partition of $2g-2$. Then there is a subspace $\mathcal{H}\mathcal{O}\mathcal{L}_g^\lambda[2]$ consisting of equivalence classes such that ω has exactly n zeros with multiplicities prescribed by λ . We now have

$$\mathcal{H}\mathcal{O}\mathcal{L}_g[2] = \coprod_{\lambda \vdash 2g-2} \mathcal{H}\mathcal{O}\mathcal{L}_g^\lambda[2].$$

Let $P(2g-2)$ denote the set of partitions of $2g-2$. The elements of $P(2g-2)$ are partially ordered by refinement and under this order the partition $[2g-2]$ is the

maximal element. Let $\overline{\mathcal{H}\mathcal{O}\mathcal{L}}_g^\lambda[2]$ denote the closure of $\mathcal{H}\mathcal{O}\mathcal{L}_g^\lambda[2]$ in $\mathcal{H}\mathcal{O}\mathcal{L}_g[2]$. Then

$$\overline{\mathcal{H}\mathcal{O}\mathcal{L}}_g^\lambda[2] = \coprod_{\lambda' \in [\lambda, [2g-2]]} \mathcal{H}\mathcal{O}\mathcal{L}_g^{\lambda'}[2].$$

The strata $\mathcal{H}\mathcal{O}\mathcal{L}_g^\lambda$ are not connected in general and Kontsevich and Zorich [KZ03] have given a complete description of their connected components. In genus 3, the result is exceptionally simple (since there are no effective even theta characteristics in genus 3). More precisely, the strata $\mathcal{H}\mathcal{O}\mathcal{L}_g^\lambda$ are connected for all λ different from $[2, 2]$ and $[4]$. In these two cases, strata decomposes as

$$\mathcal{H}\mathcal{O}\mathcal{L}_3^\lambda = \mathcal{C}_3^{\lambda, \text{h}} \sqcup \mathcal{C}_3^{\lambda, \text{q}},$$

where $\mathcal{C}_3^{\lambda, \text{h}}$ is the component whose underlying curves are hyperelliptic and $\mathcal{C}_3^{\lambda, \text{q}}$ is the component whose underlying curves are not hyperelliptic. For a more detailed discussion, see [LM14].

We introduce corresponding loci in $\mathcal{H}\mathcal{O}\mathcal{L}_3[2]$ which we denote by $\mathcal{C}_3^{\lambda, \text{h}}[2]$ and $\mathcal{C}_3^{\lambda, \text{q}}[2]$, respectively, and we denote their closures in $\mathcal{H}\mathcal{O}\mathcal{L}_3[2]$ by $\overline{\mathcal{C}}_3^{\lambda, \text{h}}[2]$ and $\overline{\mathcal{C}}_3^{\lambda, \text{q}}[2]$. However, these loci are not connected - we shall shortly see that the locus $\mathcal{C}_3^{\lambda, \text{h}}[2]$ consists of 36 connected components while the locus $\overline{\mathcal{C}}_3^{\lambda, \text{q}}[2]$ consists of 28 components. This is in sharp contrast with the strata $\mathcal{H}\mathcal{O}\mathcal{L}_g^{[2, 1^2]}[2]$ and $\mathcal{H}\mathcal{O}\mathcal{L}_g^{[3, 1]}[2]$ which remain connected after adding the level 2 structure.

5.1. Moduli of canonical divisors. There is a close connection between the moduli spaces $\mathcal{H}\mathcal{O}\mathcal{L}_3[2]$ and $\mathcal{M}_{3,1}[2]$ which we shall now explain. If ω is a nonzero holomorphic differential on a curve C and c is a nonzero constant, then $c\omega$ also is a nonzero holomorphic differential and $c\omega$ has the same zeros as ω . We may thus projectivize $\mathcal{H}\mathcal{O}\mathcal{L}_3[2]$ and the stratification of $\mathcal{H}\mathcal{O}\mathcal{L}_3[2]$ induces a stratification of $\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3[2])$. The space $\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3[2])$ parametrizes genus 3 curves with level 2 structure marked with a canonical divisor.

Now consider the locus $\mathcal{H}\mathcal{Y}\mathcal{P}_{3,1}[2] \subset \mathcal{M}_{3,1}[2]$. Let P be the marked point P of a hyperelliptic curve C . There is then a unique canonical divisor containing P in its support. If P is a Weierstrass point, then this divisor is namely $4P$ and if P is not a Weierstrass point, then this divisor is $2P + 2i(P)$ where i denotes the hyperelliptic involution. We thus see that

$$\mathbb{P}(\overline{\mathcal{C}}_3^{[2, 2], \text{h}}[2]) \cong \langle i \rangle \setminus \mathcal{H}\mathcal{Y}\mathcal{P}_{3,1}[2],$$

where $\langle i \rangle$ is the group generated by the hyperelliptic involution.

We now instead consider the locus $\mathcal{Q}_1[2] \subset \mathcal{M}_{3,1}[2]$. If P is the marked point of a plane quartic curve C we may naturally define a canonical divisor on C as $D = T_P C \cdot C$. If P is not a genuine bitangent point (i.e. a bitangent point which is not a hyperflex point), P is the unique point giving the canonical divisor D . We thus have isomorphisms

$$\begin{aligned} \mathcal{Q}_{\text{ord}}[2] &\cong \mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2, 1^2]}[2]), \\ \mathcal{Q}_{\text{fix}}[2] &\cong \mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[3, 1]}[2]), \\ \mathcal{Q}_{\text{hff}}[2] &\cong \mathbb{P}(\mathcal{C}_3^{[4], \text{q}}[2]), \\ \mathcal{Q}_{\text{ord}}[2] &\cong \mathbb{P}(\overline{\mathcal{H}\mathcal{O}\mathcal{L}}_3^{[2, 1^2]}[2]). \end{aligned}$$

However, if P is a genuine bitangent point we have that $D = T_P C \cdot C = 2P + 2Q$ for some point $Q \neq P$. Thus both P and Q give the same canonical divisor D . Let β be the involution on $\mathcal{Q}_{\text{btg}}[2]$ sending a curve marked with a bitangent point to the same curve marked with the other point sharing the same bitangent line. Then

$$\begin{aligned} \langle \beta \rangle \setminus \mathcal{Q}_{\text{btg}}[2] &\cong \mathbb{P}(\mathcal{C}_3^{[2,2],q}[2]), \\ \langle \beta \rangle \setminus \mathcal{Q}_{\text{btg}}[2] &\cong \mathbb{P}(\overline{\mathcal{C}}_3^{[2,2],q}[2]), \end{aligned}$$

where $\langle \beta \rangle$ is the group generated by β .

We have that

$$\mathcal{Q}_{\text{btg}}[2] \cong \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} T_{E_6}(e)$$

and β acts by sending $\chi \in T_{E_6}(e)$ to $\chi^{-1} \in T_{E_6}(e)$. Recall that $W(E_7) \cong \text{Sp}(6, \mathbb{F}_2) \times \mathbb{Z}_2$. The subgroup $W(E_6) \subset W(E_7)$ is entirely contained in $\text{Sp}(6, \mathbb{F}_2)$ and we may therefore identify the group generated by $W(E_6)$ and β with $W(E_6) \times \mathbb{Z}_2$. In order to compute the cohomology of $\mathbb{P}(\mathcal{C}_3^{[2,2],q}[2])$ we thus want to compute the $W(E_6) \times \mathbb{Z}_2$ -equivariant cohomology of T_{E_6} , induce up to $W(E_7)$ and then take $\{\pm 1\}$ -invariants

$$H^i(\mathbb{P}(\mathcal{C}_3^{[2,2],q}[2])) = \text{Ind}_{W(E_6) \times \mathbb{Z}_2}^{W(E_7)} (H^i(T_{E_6}))^{\{\pm 1\}}.$$

Using the results from [Ber16b], this computation is straightforward. We present the result in Table 7. In order to obtain the cohomology of $\mathbb{P}(\overline{\mathcal{C}}_3^{[2,2],q}[2])$ we use the $\text{Sp}(6, \mathbb{F}_2)$ -equivariant exact sequence of mixed Hodge structures

$$0 \rightarrow H^i(\mathbb{P}(\overline{\mathcal{C}}_3^{[2,2],q}[2])) \rightarrow H^i(\mathbb{P}(\mathcal{C}_3^{[2,2],q}[2])) \rightarrow H^{i-1}(\mathbb{P}(\mathcal{C}_3^{[4],q}[2]))(-1) \rightarrow 0.$$

The result is given in Table 8.

5.2. Cohomology of moduli spaces of Abelian differentials. Before we conclude this section we explain how to obtain the cohomology of the non-projectivized spaces from the cohomology of their projectivized counterparts.

Proposition 5.1. *The cohomology of $\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2,1^2]}[2]$ is given by*

$$H^i(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2,1^2]}[2]) = H^i(\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2,1^2]}[2])) \oplus H^{i-2}(\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2,1^2]}[2]))(-1).$$

Proof. The moduli space $\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2,1^2]}[2]$ is a \mathbb{P}^1 -fibration over $\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[2,1^2]}[2])$ and the corresponding Leray-Serre spectral sequence degenerates at the second page. One then obtains the result by reading off the diagonals. \square

Completely analogous arguments gives the following result.

Proposition 5.2.

$$\begin{aligned} H^i(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[3,1]}[2]) &= H^i(\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[3,1]}[2])) \oplus H^{i-2}(\mathbb{P}(\mathcal{H}\mathcal{O}\mathcal{L}_3^{[3,1]}[2]))(-1) \\ H^i(\mathcal{C}_3^{[2,2],q}[2]) &= H^i(\mathbb{P}(\mathcal{C}_3^{[2,2],q}[2])) \oplus H^{i-2}(\mathbb{P}(\mathcal{C}_3^{[2,2],q}[2]))(-1) \\ H^i(\mathcal{C}_3^{[4],q}[2]) &= H^i(\mathbb{P}(\mathcal{C}_3^{[4],q}[2])) \oplus H^{i-2}(\mathbb{P}(\mathcal{C}_3^{[4],q}[2]))(-1) \\ H^i(\overline{\mathcal{H}\mathcal{O}\mathcal{L}}_3^{[2,1^2]}[2]) &= H^i(\mathbb{P}(\overline{\mathcal{H}\mathcal{O}\mathcal{L}}_3^{[2,1^2]}[2])) \oplus H^{i-2}(\mathbb{P}(\overline{\mathcal{H}\mathcal{O}\mathcal{L}}_3^{[2,1^2]}[2]))(-1) \\ H^i(\overline{\mathcal{C}}_3^{[2,2],q}[2]) &= H^i(\mathbb{P}(\overline{\mathcal{C}}_3^{[2,2],q}[2])) \oplus H^{i-2}(\mathbb{P}(\overline{\mathcal{C}}_3^{[2,2],q}[2]))(-1) \end{aligned}$$

6. PLAIN PLANE QUARTICS

We now return to the moduli space $\mathcal{Q}[2]$ and compute its cohomology as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$. A step in this direction was taken in [Ber16c] where the cohomology was computed as a representation of the symmetric group on 7 elements (a subgroup of index 288 in $\mathrm{Sp}(6, \mathbb{F}_2)$ which can be thought of as the stabilizer of an unordered Aronhold set of bitangents). We only reproduce the Poincaré polynomial here and refer to the original article for the full result.

Proposition 6.1 ([Ber16c]). *The Poincaré polynomial of $\mathcal{Q}[2]$ is*

$$P(\mathcal{Q}[2], t) = 1 + 35t + 490t^2 + 3485t^3 + 13174t^4 + 24920t^5 + 18375t^6.$$

In order to continue the pursuit of the full structure as a $\mathrm{Sp}(6, \mathbb{F}_2)$ -representation we shall relate $\mathcal{Q}[2]$ to some of the spaces that have occurred elsewhere in the paper.

Lemma 6.2. *The cohomology group $H^i(\mathcal{Q}[2])$ is a subrepresentation of the $\mathrm{Sp}(6, \mathbb{F}_2)$ -representation $H^i(\mathcal{Q}_{\overline{\mathrm{btg}}}[2])$. In particular, it is pure of Tate type (i, i) .*

Proof. The forgetful morphism

$$f : \mathcal{Q}_{\overline{\mathrm{btg}}}[2] \rightarrow \mathcal{Q}[2],$$

is finite so the map

$$f_! \circ f^* : H^i(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]) \rightarrow H^i(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]),$$

is multiplication with $\deg(f) = 56$. Thus, since we are using cohomology with rational coefficients, the map

$$f^* : H^i(\mathcal{Q}[2]) \rightarrow H^i(\mathcal{Q}_{\overline{\mathrm{btg}}}[2])$$

is injective. \square

Unfortunately, the cohomology of $\mathcal{Q}_{\overline{\mathrm{btg}}}[2]$ is much too large in comparison with the cohomology of $\mathcal{Q}[2]$ for the above lemma to give any clues as it stands. The cohomology of $\mathcal{Q}_{\mathrm{fix}}[2]$ is however much smaller. To make the comparison, the following lemma shall be useful.

Lemma 6.3 (Looijenga, [Loo93]). *Let X be a variety of pure dimension and let $Y \subset X$ be a hypersurface. Then there is a Gysin exact sequence of mixed Hodge structures*

$$\cdots \rightarrow H^{k-2}(Y)(-1) \rightarrow H^k(X) \rightarrow H^k(X \setminus Y) \rightarrow H^{k-1}(Y)(-1) \rightarrow \cdots$$

Proposition 6.4. *The cohomology group $H^i(\mathcal{Q}[2])$ is a subrepresentation of the $\mathrm{Sp}(6, \mathbb{F}_2)$ -representation $H^i(\mathcal{Q}_{\mathrm{fix}}[2])$.*

Proof. Let $X = \mathcal{Q}_{\mathrm{fix}}[2] \sqcup \mathcal{Q}_{\mathrm{hfl}}[2]$ and let $Y = \mathcal{Q}_{\mathrm{hfl}}[2]$ and apply Lemma 6.3 to see that $H^i(X)$ consists of one part of Tate type (i, i) coming from $H^i(\mathcal{Q}_{\mathrm{fix}}[2])$ and one part of Tate type $(i-1, i-1)$ coming from $\mathcal{Q}_{\mathrm{hfl}}[2]$.

The morphism $f : X \rightarrow \mathcal{Q}[2]$ forgetting the marked point is finite of degree 24 so the map

$$f_! \circ f^* : H^i(X) \rightarrow H^i(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]),$$

is multiplication with 24. In particular

$$f^* : H^i(\mathcal{Q}[2]) \rightarrow H^i(X)$$

is injective. But $H^i(\mathcal{Q}[2])$ is pure of Tate type (i, i) so the image of f^* must lie inside the (i, i) -part of $H^i(X)$ which we can identify with a subspace of $H^i(\mathcal{Q}_{\text{fix}}[2])$ by the above. \square

One could now hope that knowing that $H^i(\mathcal{Q}[2])$ is a subrepresentation of $H^i(\mathcal{Q}_{\text{fix}}[2])$ together with the information about how this representation restricts to S_7 from [Ber16c] would determine $H^i(\mathcal{Q}[2])$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$. This is the case for $i = 0, 1, 2$ and 3 but not for $i = 4, 5$ and 6 . For instance, in the case $i = 4$ there are 1039 representations that satisfy these conditions.

Observe that the space $\mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2])$ parametrizes plane quartics with level 2 structure marked with a bitangent line (here we also allow hyperflex lines as bitangent lines). We consider a level 2 structure on a quartic C as an ordered Aronhold set $(\theta_1, \dots, \theta_7)$ of odd theta characteristics. The odd theta characteristics not in the Aronhold set are of the form

$$\theta_{i,j} := \sum_{k=1}^7 \theta_k - \theta_i - \theta_j, \quad 1 \leq i < j \leq 7.$$

We define

$$\mathcal{B}_i \subset \mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2])$$

as the subset of points such that the marked bitangent induces the i 'th theta characteristic of the ordered Aronhold set. Similarly, we define $\mathcal{B}_{i,j}$ as the subset where the marked bitangent induces the theta characteristic $\theta_{i,j}$.

Let I be any subset of $\{1, \dots, 7\}$ of size 1 or 2. The spaces \mathcal{B}_I are all isomorphic and

$$\mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2]) = \coprod_I \mathcal{B}_I$$

where the union is over all such subsets I . Moreover, we have an isomorphism

$$\mathcal{Q}[2] \rightarrow \mathcal{B}_1$$

sending the class of a plane quartic with level 2 structure, where we think of the level structure as an ordered Aronhold set of bitangent lines, to the class of the same curve with the same level 2 structure marked with the first bitangent of the Aronhold set.

We now rephrase the above slightly. It is well-known that the stabilizer $\text{Stab}(b) \subset \text{Sp}(6, \mathbb{F}_2)$ of a bitangent line b is isomorphic to $W(E_6)$. Let \mathcal{S} denote the quotient set $\text{Sp}(6, \mathbb{F}_2)/W(E_6)$ and let $[\sigma]$ denote the class of $\sigma \in \text{Sp}(6, \mathbb{F}_2)$ in \mathcal{S} . If we now let $X_{[\sigma]} = \mathcal{Q}[2]$ we have

$$\mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2]) \cong \coprod_{\sigma \in \mathcal{S}} X_{[\sigma]}$$

and $\text{Sp}(6, \mathbb{F}_2)$ acts transitively on the set of connected components of $\mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2])$. In particular, we find that

$$(6.1) \quad H^i(\mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2])) = \text{Ind}_{W(E_6)}^{\text{Sp}(6, \mathbb{F}_2)} \text{Res}_{W(E_6)}^{\text{Sp}(6, \mathbb{F}_2)} (H^i(\mathcal{Q}[2])).$$

We now know how the cohomology groups of $\mathcal{Q}[2]$ restricts to S_7 , and we know how they relate both to the cohomology groups of $\mathcal{Q}_{\text{fix}}[2]$ and the cohomology groups of $\mathbb{P}(\bar{\mathcal{C}}_3^{[2,2],q}[2])$. This information is enough to finally determine $H^i(\mathcal{Q}[2])$ in all degrees. The results are presented in Table 9. We remark that even though it

may seem plausible at first glance, Equation 6.1 does not determine the cohomology of $\mathcal{Q}[2]$ by itself.

7. THE MODULI SPACE $\mathcal{M}_3[2]$

We now consider the cohomology of the moduli space $\mathcal{M}_3[2]$ of genus 3 curves with level 2 structure. By applying Lemma 6.3 to the decomposition $\mathcal{M}_3[2] = \mathcal{Q}[2] \sqcup \mathcal{Hyp}_3[2]$ we obtain the exact sequence of mixed Hodge structures

$$\cdots \rightarrow H^{k-2}(\mathcal{Hyp}_3[2])(-1) \rightarrow H^k(\mathcal{M}_3[2]) \rightarrow H^k(\mathcal{Q}[2]) \rightarrow H^{k-1}(\mathcal{Hyp}_3[2])(-1) \rightarrow \cdots$$

Since both $H^k(\mathcal{Hyp}_3[2])(-1)$ and $H^k(\mathcal{Q}[2])(-1)$ are pure of Tate type (k, k) , the above sequence decomposes into sequences

$$0 \rightarrow W_{2k}H^k(\mathcal{M}_3[2]) \rightarrow H^k(\mathcal{Q}[2]) \rightarrow H^{k-1}(\mathcal{Hyp}_3[2])(-1) \rightarrow W_{2k}H^{k+1}(\mathcal{M}_3[2]) \rightarrow 0,$$

where $W_{2k}H^k(\mathcal{M}_3[2])$ denotes the weight $2k$ part of $H^k(\mathcal{M}_3[2])$. Taking $k = 0$ we obtain that $H^0(\mathcal{M}_3[2]) = H^0(\mathcal{Q}[2]) = \mathbb{Q}$ which is not very surprising.

For $k = 1$ and $k = 2$ we have the results of Hain [Hai95] and Putman [Put12] that $H^k(\mathcal{M}_3[2]) \cong H^k(\mathcal{M}_3)$. Combined with the results of Looijenga [Loo93] we have that $H^1(\mathcal{M}_3[2]) = 0$ and $H^2(\mathcal{M}_3[2]) = \mathbb{Q}$ with Tate type $(1, 1)$ and it is reassuring to see that this is indeed compatible with the above sequence.

Given the complexity of the cohomology of $\mathcal{Q}[2]$ and $\mathcal{Hyp}_3[2]$, the cohomology of $\mathcal{M}_3[2]$ is surprisingly simple in low degrees. This phenomenon will not prevail in all degrees though. For instance, taking $k = 7$ the above sequence gives that $\dim(H^7(\mathcal{M}_3[2])) \geq 7680$. This bound is in fact far from optimal, as Fullarton and Putman [FP16] recently have shown that $\dim(H^7(\mathcal{M}_3[2])) \geq 11520$ via completely different methods. In particular, we see that the cohomology of $\mathcal{M}_3[2]$ is not the smallest possible fitting in a four term exact sequence of the above type. We also remark that we get an upper bound $\dim(H^7(\mathcal{M}_3[2])) \leq \dim(H^5(\mathcal{Hyp}_3[2])) = 25920$.

7.1. The weighted Euler characteristic. Recall that the Poincaré-Serre polynomial of a variety X is defined as

$$PS(X, t, u) = \sum_{i,j \geq 0} W_j H^i(X) t^i u^j$$

where the sum is taken in the Grothendieck ring of vector spaces. By setting $t = -1$ in $PS(X, t, u)$ we obtain the weighted Euler characteristic $\text{Eul}(X, u)$. Using the above exact sequence the weighted Euler characteristic of $\mathcal{M}_3[2]$ can easily be deduced from Table 9 and Table 10.

8. TABLES

In the tables below we present the cohomology of various spaces occurring throughout the paper as representations of the group $\mathrm{Sp}(6, \mathbb{F}_2)$. The rows of the tables represent the cohomology groups and the columns correspond to irreducible representations of $\mathrm{Sp}(6, \mathbb{F}_2)$. Thus, a number n in the row indexed by H^i and column indexed by ϕ means that the irreducible representation ϕ occurs with multiplicity n in H^i .

The irreducible representations are denoted as ϕ_{dx} where the subscripts follow the conventions of [CCN⁺85], i.e. d denotes the dimension of the representation and the letter x denotes is used to distinguish different representations of the same dimension. The letters used here are the same as in [CCN⁺85].

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	1	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	2	0	2	0	0
H^3	0	0	0	3	0	3	0	3	0	0
H^4	0	0	0	7	0	8	2	9	0	5
H^5	0	0	3	17	2	25	16	30	11	30
H^6	2	4	18	34	19	50	45	63	53	86
H^7	2	8	19	34	25	43	47	52	74	101
<hr/>										
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	1	0	2	1	0	0	0	0
H^3	1	0	7	2	9	7	5	0	0	4
H^4	9	1	27	14	33	36	33	5	7	32
H^5	50	29	78	63	99	128	125	61	73	128
H^6	127	113	160	154	194	267	277	215	233	295
H^7	117	137	159	147	185	249	276	255	251	307
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	2	0	0	2	0	0	0	0	0	0
H^3	13	1	0	9	0	2	2	11	7	6
H^4	51	13	19	47	21	33	33	73	61	61
H^5	157	99	126	191	141	179	188	268	258	290
H^6	326	287	351	427	393	456	498	588	598	710
H^7	313	296	388	404	441	468	533	598	602	731

TABLE 1. The cohomology of $\mathcal{Q}_{\text{ord}}[2]$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	0	0	1	0	1	0	0
H^3	0	0	0	2	0	2	0	2	0	0
H^4	0	0	0	5	0	6	1	6	0	5
H^5	0	0	3	10	2	15	11	18	9	20
H^6	1	2	7	13	8	17	18	22	23	35
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	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	1	0	2	1	0	0	0	2
H^3	1	0	6	2	6	6	4	0	0	10
H^4	8	1	18	11	24	27	25	5	7	35
H^5	32	22	45	39	59	76	77	45	51	93
H^6	47	47	60	58	69	98	104	88	92	120
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	0	2	0	0	0	0	0	0
H^3	4	1	0	7	0	2	2	10	7	6
H^4	25	12	17	36	19	27	29	55	47	50
H^5	80	68	83	118	96	116	124	164	160	184
H^6	111	111	140	157	155	175	193	221	226	272

TABLE 2. The cohomology of $\mathcal{Q}_{\text{fix}}[2]$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	1	0	0	1	0	3	0	2	0	0
H^2	0	0	0	2	0	4	0	5	0	0
H^3	0	0	0	6	0	7	2	10	0	4
H^4	0	0	3	17	2	20	15	25	11	30
H^5	1	4	14	30	17	41	39	49	51	80
H^6	2	7	18	25	22	35	39	44	60	78
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	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	2	1	0	0	0	1
H^2	1	0	6	2	9	8	4	0	0	11
H^3	10	1	24	15	30	35	31	4	7	48
H^4	46	27	72	60	89	114	118	58	69	146
H^5	105	105	140	129	169	229	243	198	206	282
H^6	98	112	124	119	143	197	215	207	207	244
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	3	1	0	10	0	2	1	9	5	5
H^3	27	14	16	49	18	32	30	67	56	58
H^4	120	92	120	173	134	168	179	252	242	272
H^5	265	250	319	364	360	401	447	522	526	629
H^6	241	243	309	323	349	375	423	463	477	578

TABLE 3. The cohomology of $\mathcal{Q}_{\text{btg}}[2]$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	0	0	0	0	0	1	0	2	0	0
H^2	0	0	0	0	0	1	0	2	0	0
H^3	0	0	0	4	0	3	1	3	0	3
H^4	0	0	1	6	1	7	5	7	6	13
H^5	0	1	4	5	4	8	9	11	10	14
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	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	1	1	0	0	0	1
H^2	1	0	3	2	4	4	2	0	0	6
H^3	5	1	10	7	13	15	16	3	5	21
H^4	15	12	25	19	31	39	41	26	26	49
H^5	23	22	26	27	31	45	46	40	44	53
<hr/>										
	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	2	1	0	7	0	2	1	6	4	4
H^3	15	8	10	21	12	17	19	33	29	31
H^4	44	34	48	57	54	60	68	90	85	99
H^5	49	53	62	74	69	81	86	96	102	122

TABLE 4. The cohomology of $\mathcal{Q}_{\text{hff}}[2]$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	1	0	1	0	0
H^3	0	0	0	3	0	2	0	2	0	0
H^4	0	0	0	5	0	6	2	7	0	5
H^5	0	0	3	12	2	19	15	24	11	25
H^6	2	4	15	24	17	35	34	45	44	66
H^7	1	6	12	21	17	26	29	30	51	66
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	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	1	0	2	1	0	0	0	0
H^3	1	0	6	2	7	6	5	0	0	2
H^4	8	1	21	12	27	30	29	5	7	22
H^5	42	28	60	52	75	101	100	56	66	93
H^6	95	91	115	115	135	191	200	170	182	202
H^7	70	90	99	89	116	151	172	167	159	187
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	2	0	0	2	0	0	0	0	0	0
H^3	13	1	0	7	0	2	2	11	7	6
H^4	47	12	19	40	21	31	31	63	54	55
H^5	132	87	109	155	122	152	159	213	211	240
H^6	246	219	268	309	297	340	374	424	438	526
H^7	202	185	248	247	286	293	340	377	376	459

TABLE 5. The cohomology of $\mathcal{Q}_{\text{ord}}^{\text{---}}[2]$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	0	0	0	1	0	2	0	2	0	0
H^2	0	0	0	2	0	3	0	3	0	0
H^3	0	0	0	6	0	6	2	8	0	4
H^4	0	0	3	13	2	17	14	22	11	27
H^5	1	4	13	24	16	34	34	42	45	67
H^6	2	6	14	20	18	27	30	33	50	64
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	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	2	1	0	0	0	1
H^2	1	0	5	2	8	7	4	0	0	10
H^3	9	1	21	13	26	31	29	4	7	42
H^4	41	26	62	53	76	99	102	55	64	125
H^5	90	93	115	110	138	190	202	172	180	233
H^6	75	90	98	92	112	152	169	167	163	191
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	3	1	0	9	0	2	1	9	5	5
H^3	25	13	16	42	18	30	29	61	52	54
H^4	105	84	110	152	122	151	160	219	213	241
H^5	221	216	271	307	306	341	379	432	441	530
H^6	192	190	247	249	280	294	337	367	375	456

TABLE 6. The cohomology of $\mathcal{Q}_{\text{btg}}^{\text{---}}[2]$ as a representation of $\text{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	1	0	0	0	0	2	0	2	0	0
H^2	0	0	0	1	0	2	0	4	0	0
H^3	0	0	0	5	0	4	1	5	0	3
H^4	0	0	1	11	1	11	8	12	8	20
H^5	1	2	8	16	9	23	22	28	28	44
H^6	2	3	14	11	14	21	23	31	26	34
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	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	1	1	0	0	0	1
H^2	1	0	4	2	5	5	3	0	0	8
H^3	6	1	14	10	19	20	21	3	5	31
H^4	23	17	44	31	54	65	70	36	38	86
H^5	58	58	79	69	93	128	133	109	112	155
H^6	65	59	61	73	70	110	111	106	121	129
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	2	1	0	8	0	2	1	6	4	4
H^3	18	9	11	30	13	21	22	44	37	38
H^4	75	50	74	95	85	96	108	155	143	161
H^5	147	137	177	200	199	220	244	288	290	346
H^6	117	143	157	186	169	207	217	225	253	303

TABLE 7. The cohomology of $\mathbb{P}(\mathcal{C}_3^{[2,2],q}[2])$ as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	0	0	0	0	0	1	0	2	0	0
H^2	0	0	0	1	0	1	0	2	0	0
H^3	0	0	0	5	0	3	1	3	0	3
H^4	0	0	1	7	1	8	7	9	8	17
H^5	1	2	7	10	8	16	17	21	22	31
H^6	2	2	10	6	10	13	14	20	16	20
<hr/>										
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	1	1	0	0	0	1
H^2	1	0	3	2	4	4	3	0	0	7
H^3	5	1	11	8	15	16	19	3	5	25
H^4	18	16	34	24	41	50	54	33	33	65
H^5	43	46	54	50	62	89	92	83	86	106
H^6	42	37	35	46	39	65	65	66	77	76
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	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	2	1	0	7	0	2	1	6	4	4
H^3	16	8	11	23	13	19	21	38	33	34
H^4	60	42	64	74	73	79	89	122	114	130
H^5	103	103	129	143	145	160	176	198	205	247
H^6	68	90	95	112	100	126	131	129	151	181

TABLE 8. The cohomology of $\mathbb{P}(\overline{\mathcal{C}}_3^{[2,2],q}[2])$ as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	1	0	0
H^2	0	0	0	0	0	0	0	0	0	0
H^3	0	0	0	1	0	0	0	0	0	0
H^4	0	0	0	0	0	0	0	0	0	1
H^5	0	0	0	0	0	1	1	1	0	0
H^6	1	0	2	0	1	1	1	3	0	0
<hr/>										
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	0	0	0	0	0	0	0	1
H^3	0	0	1	0	0	0	1	0	0	2
H^4	0	0	2	0	2	1	2	1	0	3
H^5	1	2	2	1	2	4	3	3	3	4
H^6	5	1	1	4	0	3	2	2	5	3
<hr/>										
	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	0	1	0	0	0	0	0	0
H^3	1	0	0	0	0	0	1	2	2	1
H^4	4	0	3	1	3	2	3	6	5	4
H^5	4	4	4	6	5	6	6	6	8	9
H^6	1	6	3	6	1	6	4	2	6	6

TABLE 9. The cohomology of $\mathcal{Q}[2]$ as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	1	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	0	0	0	0	0
H^3	0	0	0	1	0	0	0	0	0	1
H^4	0	0	0	0	0	1	1	1	0	1
H^5	0	0	1	0	1	1	1	2	0	0
<hr/>										
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	0	0	1
H^2	0	0	1	0	2	1	1	0	0	3
H^3	0	0	3	1	3	2	4	1	0	5
H^4	2	2	3	2	3	6	5	4	4	6
H^5	4	2	1	4	1	4	3	3	6	4
<hr/>										
	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	1	0	0	2	0	0	1	3	2	2
H^3	5	1	3	3	4	3	5	10	7	7
H^4	6	5	7	8	7	9	9	10	12	14
H^5	2	7	4	8	3	8	6	4	8	9

TABLE 10. The cohomology of $\mathcal{H}yp_3[2]$ as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$.

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